

EXCEPTIONAL SETS FOR THE DERIVATIVES OF BLASCHKE PRODUCTS

EMMANUEL FRICAIN, JAVAD MASHREGHI

ABSTRACT. We obtain growth estimates for the logarithmic derivative $B'(z)/B(z)$ of a Blaschke product as $|z| \rightarrow 1$ and z avoids some exceptional sets.

1. INTRODUCTION

Let f be a meromorphic function in the unit disc \mathbb{D} . Then its order is defined by

$$\sigma = \limsup_{r \rightarrow 1^-} \frac{\log^+ T(r)}{\log 1/(1-r)},$$

where

$$T(r) = \frac{1}{\pi} \int_{\{|z| < r\}} \frac{|f'(z)|^2}{(1 + |f(z)|^2)^2} \log\left(\frac{r}{|z|}\right) dx dy$$

is the Nevanlinna characteristic of f [13]. Meromorphic functions of finite order have been extensively studied and they have numerous applications in pure and applied mathematics, e.g. in linear differential equations. In many applications a major role is played by the logarithmic derivative of meromorphic functions and we need to obtain sharp estimates for the logarithmic derivative as we approach to the boundary [7, 8]. In particular, the following result for the rate of growth of meromorphic functions of finite order in the unit disc has application in the study of linear differential equations [10, Theorem 5.1].

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Theorem 1.1. *Let f be a meromorphic function in the unit disc \mathbb{D} of finite order σ and let $\varepsilon > 0$. Then the following two statements hold.*

(a) *There exists a set $E_1 \subset (0, 1)$ which satisfies*

$$\int_{E_1} \frac{dr}{1-r} < \infty,$$

such that, for all $z \in \mathbb{D}$ with $|z| \notin E_1$, we have

$$(1.1) \quad \left| \frac{f'(z)}{f(z)} \right| \leq \frac{1}{(1-|z|)^{3\sigma+4+\varepsilon}}.$$

(b) *There exists a set $E_2 \subset [0, 2\pi)$ whose Lebesgue measure is zero and a function*

$R(\theta) : [0, 2\pi) \setminus E_2 \longrightarrow (0, 1)$ such that for all $z = re^{i\theta}$ with $\theta \in [0, 2\pi) \setminus E_2$ and $R(\theta) < r < 1$ the inequality (1.1) holds.

Clearly, the relation (1.1) can also be written as

$$\left| \frac{f'(z)}{f(z)} \right| = \frac{O(1)}{(1-|z|)^{3\sigma+4+\varepsilon}}$$

as $|z| \rightarrow 1$. But we should note that in case (b) it does not hold uniformly with respect to $|z|$.

Let $(z_n)_{n \geq 1}$ be a sequence in the unit disc satisfying the Blaschke condition

$$(1.2) \quad \sum_{n=1}^{\infty} (1-|z_n|) < \infty.$$

Then the Blaschke product

$$B(z) = \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \bar{z}_n z}$$

is an analytic function in the unit disc with order $\sigma = 0$ and

$$(1.3) \quad \frac{B'(z)}{B(z)} = \sum_{n=1}^{\infty} \frac{1-|z_n|^2}{(1-\bar{z}_n z)(z-z_n)}.$$

Thus Theorem 1.1 implies that, for any $\varepsilon > 0$,

$$\left| \sum_{n=1}^{\infty} \frac{1-|z_n|^2}{(1-\bar{z}_n z)(z-z_n)} \right| = \frac{O(1)}{(1-|z|)^{4+\varepsilon}}$$

as $|z| \rightarrow 1^-$ in any of the two manners explained above. In this paper, instead of (1.2), we pose more restrictive conditions on the rate of convergence of zeros z_n and instead we improve the exponent $4 + \varepsilon$. The most common condition is

$$(1.4) \quad \sum_{n=1}^{\infty} (1 - |z_n|)^{\alpha} < \infty,$$

for some $\alpha \in (0, 1]$. However, we consider a more general assumption

$$(1.5) \quad \sum_{n=1}^{\infty} h(1 - |z_n|) < \infty,$$

where h is a positive continuous function satisfying certain smoothness conditions which will be described below. Our main prototype for h is

$$(1.6) \quad h(t) = t^{\alpha} (\log 1/t)^{\alpha_1} (\log_2 1/t)^{\alpha_2} \cdots (\log_n 1/t)^{\alpha_n},$$

where $\log_n = \log \log \cdots \log$ (n times), $\alpha \in (0, 1]$ and $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}$. If $\alpha = 1$ the first nonzero exponent among $\alpha_1, \alpha_2, \dots, \alpha_n$ is positive [12].

The function h is usually defined in an open interval $(0, \epsilon)$. Of course, by extending its domain of definition, we may assume that h is defined on the interval $(0, 1)$, or if required, on the entire positive real axis. Moreover, since a Blaschke sequence satisfies (1.2), the condition (1.5) will provide further information about the rate of increase of the zeros provided that $h(t) \geq C t$ as $t \rightarrow 0$.

The condition (1.4) has been extensively studied by many authors [1, 2, 3, 9, 11, 14] to obtain estimates for the integral means of the derivative of Blaschke products. We [6] have recently shown that many of these estimates can be generalized for Blaschke products satisfying (1.5).

2. CIRCULAR EXCEPTIONAL SETS

The function h given in (1.6) satisfies the following conditions:

- a) h is continuous, positive and increasing with $h(0+) = 0$;
- b) $h(t)/t$ is decreasing;

In the following, we just need these conditions. Hence, we state our results for a general function h satisfying *a)* and *b)*.

Theorem 2.1. *Let $(z_n)_{n \geq 1}$ be a sequence in the unit disc satisfying*

$$\sum_{n=1}^{\infty} h(1 - |z_n|) < \infty$$

and let B be the Blaschke product formed with zeros z_n , $n \geq 1$. Let $\beta \geq 1$. Then there is an exceptional set $E \subset (0, 1)$ such that

$$\int_E \frac{dt}{(1-t)^\beta} < \infty$$

and that

$$\left| \frac{B'(z)}{B(z)} \right| = \frac{o(1)}{(1-|z|)^\beta h^2(1-|z|)}$$

as $|z| \rightarrow 1^-$ with $|z| \notin E$.

Proof. Without loss of generality, assume that $h(t) < 1$ for $t \in (0, 1)$. Let

$$E = \bigcup_{n=1}^{\infty} \left(|z_n| - (1 - |z_n|)^\beta h(1 - |z_n|), |z_n| + (1 - |z_n|)^\beta h(1 - |z_n|) \right).$$

In the definition of E we implicitly assume that $|z_n| - (1 - |z_n|)^\beta h(1 - |z_n|) > 0$ in order to have $E \subset (0, 1)$. Certainly this condition holds for large values of n . If it does not hold for some small values of n , we simply remove those intervals from the definition of E .

Let $z \in \mathbb{D}$ with $|z| \notin E$ and fix $0 < \delta \leq (1 - |z|)/2$. By (1.3), we have

$$\frac{B'(z)}{B(z)} = \left(\sum_{|z| - |z_n| \geq \delta} + \sum_{|z| - |z_n| < \delta} \right) \frac{1 - |z_n|^2}{(1 - \bar{z}_n z)(z - z_n)}.$$

We use different techniques to estimate each sum. For the first sum we have

$$\sum_{|z| - |z_n| \geq \delta} \frac{1 - |z_n|^2}{|1 - \bar{z}_n z| |z - z_n|} \leq \frac{2}{\delta} \sum_{|z| - |z_n| \geq \delta} \frac{1 - |z_n|}{1 - |z_n| |z|}.$$

But

$$\frac{1 - |z_n|}{1 - |z||z_n|} = \left(\frac{1 - |z_n|}{h(1 - |z_n|)} \frac{h(1 - |z||z_n|)}{1 - |z||z_n|} \right) \left(\frac{h(1 - |z_n|)}{h(1 - |z||z_n|)} \right).$$

Since $h(t)$ is increasing and $h(t)/t$ is decreasing, we get

$$\frac{1 - |z_n|}{1 - |z||z_n|} \leq \frac{h(1 - |z_n|)}{h(1 - |z|)}$$

and thus

$$\sum_{|z| - |z_n| \geq \delta} \frac{1 - |z_n|^2}{|1 - \bar{z}_n z||z - z_n|} \leq \frac{2 \sum_{|z| - |z_n| \geq \delta} h(1 - |z_n|)}{\delta h(1 - |z|)} \leq \frac{C}{\delta h(1 - |z|)}.$$

A generalized version of this estimation technique has been used in [6, Lemma 2.1].

To estimate the second sum, we see that

$$\begin{aligned} \left| \frac{1 - |z_n|^2}{(1 - \bar{z}_n z)(z - z_n)} \right| &\leq \frac{2}{|z - z_n|} \leq \frac{2}{(1 - |z_n|)^\beta h(1 - |z_n|)} \\ &\leq \frac{C}{(1 - |z|)^\beta h(1 - |z|)}, \end{aligned}$$

and thus

$$\left| \sum_{|z| - |z_n| < \delta} \frac{1 - |z_n|^2}{(1 - \bar{z}_n z)(z - z_n)} \right| \leq C \frac{n(|z| + \delta) - n(|z| - \delta)}{(1 - |z|)^\beta h(1 - |z|)},$$

where $n(t)$ is the number of points z_n lying in the disc $\{z : |z| \leq t\}$. Therefore

$$(2.1) \quad \left| \frac{B'(z)}{B(z)} \right| \leq \frac{C}{h(1 - |z|)} \left(\frac{1}{\delta} + \frac{n(|z| + \delta) - n(|z| - \delta)}{(1 - |z|)^\beta} \right)$$

provided that $z \in \mathbb{D}$ with $|z| \notin E$. The best choice of δ depends on the counting function $n(t)$. We make a choice for the most general case.

Assume that $\delta = (1 - |z|)/2$. Our assumption (1.5) on the rate of increase of zeros z_n is equivalent to

$$\int_0^1 h(1 - t) \, dn(t) < \infty,$$

and it is well known that this condition implies

$$(2.2) \quad n(t) = \frac{o(1)}{h(1-t)}$$

as $t \rightarrow 1^-$. Therefore,

$$(2.3) \quad n(|z| + \delta) - n(|z| - \delta) \leq \frac{o(1)}{h(1 - |z|)}.$$

Hence, by (2.1) and (2.3), we get the promised growth for B'/B . To verify the size of E , note that

$$\begin{aligned} \int_E \frac{dt}{(1-t)^\beta} &= \sum_{n=1}^{\infty} \int_{|z_n| - (1-|z_n|)^\beta h(1-|z_n|)}^{|z_n| + (1-|z_n|)^\beta h(1-|z_n|)} \frac{dt}{(1-t)^\beta} \\ &= \sum_{n=1}^{\infty} \int_{(1-|z_n|) - (1-|z_n|)^\beta h(1-|z_n|)}^{(1-|z_n|) + (1-|z_n|)^\beta h(1-|z_n|)} \frac{d\tau}{\tau^\beta} \\ &\leq \sum_{n=1}^{\infty} \frac{2(1-|z_n|)^\beta h(1-|z_n|)}{(1-|z_n|) - (1-|z_n|)^\beta h(1-|z_n|)^\beta} \\ &\leq C \sum_{n=1}^{\infty} h(1-|z_n|) < \infty. \end{aligned}$$

□

Remark 1: As the counting function $n(t) = 1/(1-t)^\alpha$ suggests, the assumption

$$(2.4) \quad n(|z| + \delta) - n(|z| - \delta) \leq C \frac{\delta n(|z|)}{1 - |z|}$$

is fulfilled by a wide class of distribution of zeros. If (2.4) holds, by (2.3) and (2.1) with

$$\delta = (1 - |z|)^{\frac{1+\beta}{2}} h^{\frac{1}{2}}(1 - |z|),$$

we obtain

$$\left| \frac{B'(z)}{B(z)} \right| = \frac{O(1)}{(1 - |z|)^{\frac{1+\beta}{2}} h^{\frac{3}{2}}(1 - |z|)}$$

as $|z| \rightarrow 1^-$ with $|z| \notin E$.

Remark 2: Let us call φ almost increasing if $\varphi(x) \leq \text{Const } \varphi(y)$ provided that $x \leq y$. Almost decreasing functions are defined similarly. As it can be easily verified, Theorem 2.1 (and also Theorem 3.1) is still true if we assume that $h(t)$ is almost increasing and $h(t)/t$ is almost decreasing.

Corollary 2.2. *Let $\alpha \in (0, 1]$, and $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}$. Let $(z_n)_{n \geq 1}$ be a sequence in the unit disc with*

$$\sum_{n=1}^{\infty} (1 - |z_n|)^{\alpha} (\log 1/(1 - |z_n|))^{\alpha_1} \dots (\log_n 1/(1 - |z_n|))^{\alpha_n} < \infty$$

and let B be the Blaschke product formed with zeros z_n , $n \geq 1$. Let $\beta \geq 1$. Then there is an exceptional set $E \subset (0, 1)$ such that

$$\int_E \frac{dt}{(1-t)^{\beta}} < \infty$$

and that

$$(2.5) \quad \left| \frac{B'(z)}{B(z)} \right| = \frac{o(1)}{(1 - |z|)^{\beta+2\alpha} (\log 1/(1 - |z|))^{2\alpha_1} \dots (\log_n 1/(1 - |z|))^{2\alpha_n}}$$

as $|z| \rightarrow 1^-$ with $|z| \notin E$.

In particular, if

$$(2.6) \quad \sum_{n=1}^{\infty} (1 - |z_n|)^{\alpha} < \infty,$$

then, for any $\beta \geq 1$, there is an exceptional set $E \subset (0, 1)$ such that

$$(2.7) \quad \int_E \frac{dt}{(1-t)^{\beta}} < \infty$$

and that

$$\left| \frac{B'(z)}{B(z)} \right| = \frac{o(1)}{(1 - |z|)^{\beta+2\alpha}}$$

as $|z| \rightarrow 1^-$ with $|z| \notin E$. If $(|z_n|)_{n \geq 1}$ is an interpolating sequence then

$$1 - |z_{n+1}| \leq c(1 - |z_n|)$$

for a constant $c < 1$ [4, Theorem 9.2]. Hence, (2.6) is satisfied for any $\alpha > 0$ and thus, for any $\beta \geq 1$ and for any $\varepsilon > 0$, there is an exceptional set E satisfying (2.7) such that

$$(2.8) \quad \left| \frac{B'(z)}{B(z)} \right| = \frac{o(1)}{(1 - |z|)^{\beta + \varepsilon}}$$

as $|z| \rightarrow 1^-$ with $|z| \notin E$. It is interesting to know if in (2.8) we are able to replace ε by zero.

3. RADIAL EXCEPTIONAL SETS

Contrary to the preceding section, we now study the behavior of

$$\left| \frac{B'(re^{i\theta})}{B(re^{i\theta})} \right|$$

as $r \rightarrow 1$ for a *fixed* θ . We obtain an upper bound for the quotient B'/B as long as $e^{i\theta} \in \mathbb{T} \setminus E$ where E is an exceptional set of Lebesgue measure zero.

Theorem 3.1. *Let B be the Blaschke product formed with zeros $z_n = r_n e^{i\theta_n}$, $n \geq 1$, satisfying*

$$\sum_{n=1}^{\infty} h(1 - r_n) < \infty.$$

Then there is an exceptional set $E \subset \mathbb{T}$ whose Lebesgue measure $|E|$ is zero such that for all $z = re^{i\theta}$ with $e^{i\theta} \in \mathbb{T} \setminus E$

$$\left| \frac{B'(z)}{B(z)} \right| = \frac{o(1)}{(1 - |z|) h(1 - |z|)}$$

as $|z| \rightarrow 1^-$.

Proof. Let us consider the open set

$$U_n = \{ z \in \mathbb{D} : (1 - |z|) > C|z - z_n| \}$$

with $C > 1$, and we define

$$I_n = \{ \zeta \in \mathbb{T} : \exists z \in U_n \text{ \& } \zeta = z/|z| \}.$$

In other words, I_n is the radial projection of U_n on the unit circle \mathbb{T} . Then we know that

$$(3.1) \quad |I_n| \leq C'(1 - r_n),$$

where C' is a constant just depending on C . Let

$$E = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} I_k.$$

By (3.1), we see that $|E| = 0$.

Fix $z \in \mathbb{D}$ with $z/|z| \notin E$. Hence, there is N such that $z/|z| \notin I_k$ for all $k \geq N$.

Let $R = (1 + |z|)/2$. Now, we write

$$\frac{B'(z)}{B(z)} = \left(\sum_{|z_n| \geq R} + \sum_{|z_n| < R, n \geq N} + \sum_{n=1}^{N-1} \right) \frac{1 - |z_n|^2}{(1 - \bar{z}_n z)(z - z_n)},$$

and as in the preceding case

$$(3.2) \quad \sum_{|z_n| \geq R} \frac{1 - |z_n|^2}{|1 - \bar{z}_n z| |z - z_n|} \leq \frac{o(1)}{(1 - |z|) h(1 - |z|)}.$$

To estimate the second sum, we see that

$$\left| \frac{1 - |z_n|^2}{(1 - \bar{z}_n z)(z - z_n)} \right| \leq \frac{2}{|z - z_n|} \leq \frac{2C}{1 - |z|}, \quad (|z| \notin E),$$

and thus, by (2.2),

$$(3.3) \quad \left| \sum_{|z_n| < R, n \geq N} \frac{1 - |z_n|^2}{(1 - \bar{z}_n z)(z - z_n)} \right| \leq \frac{2C n(R)}{1 - |z|} \leq \frac{o(1)}{(1 - |z|) h(1 - |z|)}.$$

Since the last sum is uniformly bounded (θ is fixed), (3.2) and (3.3) give the required result. \square

Corollary 3.2. *Let $\alpha \in (0, 1]$, and $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}$. If $\alpha = 1$ the first nonzero number among $\alpha_1, \alpha_2, \dots, \alpha_n$ is positive. Let B be the Blaschke product formed with*

zeros $z_n = r_n e^{i\theta_n}$, $n \geq 1$, satisfying

$$\sum_{n=1}^{\infty} (1 - r_n)^{\alpha} (\log 1/(1 - r_n))^{\alpha_1} \cdots (\log_n 1/(1 - r_n))^{\alpha_n} < \infty.$$

Then there is an exceptional set $E \subset \mathbb{T}$ whose Lebesgue measure $|E|$ is zero such that for all $z = re^{i\theta}$ with $e^{i\theta} \in \mathbb{T} \setminus E$

$$(3.4) \quad \left| \frac{B'(z)}{B(z)} \right| = \frac{o(1)}{(1 - |z|)^{1+\alpha} (\log 1/(1 - |z|))^{\alpha_1} \cdots (\log_n 1/(1 - |z|))^{\alpha_n}}$$

as $|z| \rightarrow 1^-$.

In particular, if

$$\sum_{n=1}^{\infty} (1 - r_n)^{\alpha} < \infty,$$

then there is an exceptional set $E \subset \mathbb{T}$ whose Lebesgue measure $|E|$ is zero such that for all $z = re^{i\theta}$ with $e^{i\theta} \in \mathbb{T} \setminus E$

$$(3.5) \quad \left| \frac{B'(z)}{B(z)} \right| = \frac{o(1)}{(1 - |z|)^{1+\alpha}}$$

as $|z| \rightarrow 1^-$.

Remark: Theorems 2.1 and 3.1 can be easily generalized to obtain estimates for

$$\frac{B^{(k)}(z)}{B^{(j)}(z)}$$

as $|z| \rightarrow 1^-$. This is a standard technique which can be found for example in [9, 11].

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UNIVERSITÉ DE LYON, LYON, F-69003, FRANCE; UNIVERSITÉ LYON I, INSTITUT CAMILLE JORDAN, VILLEURBANNE CEDEX F-69622, FRANCE; CNRS, UMR5208, VILLEURBANNE, F-69622, FRANCE.

E-mail address: fricain@math.univ-lyon1.fr

DÉPARTEMENT DE MATHÉMATIQUES ET DE STATISTIQUE, UNIVERSITÉ LAVAL, QUÉBEC, QC, CANADA G1K 7P4.

E-mail address: Javad.Mashreghi@mat.ulaval.ca